

# Quasi-permutable normal operators in octonion Hilbert spaces and spectra

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## Abstract

Families of quasi-permutable normal operators in octonion Hilbert spaces are investigated. Their spectra are studied. Multiparameter semigroups of such operators are considered. A non-associative analog of Stone's theorem is proved. <sup>1</sup>

## 1 Introduction

The theory of bounded and unbounded normal operators over the complex field is classical and have found many-sided applications in functional analysis, differential and partial differential equations and their applications in the sciences [4, 11, 12, 14, 32]. Nevertheless, hypercomplex analysis is fast developing, because it is closely related with problems of theoretical and mathematical physics and of partial differential equations [2, 7, 9]. On the other hand, the octonion algebra is the largest division real algebra in which the complex field has non-central embeddings [3, 1, 13]. The octonion algebra also is intensively used in mathematics and various applications [5, 10, 8, 15, 16].

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Previously analysis over quaternion and octonions was developed and spectral theory of bounded normal operators and unbounded self-adjoint operators was described [18, 19, 20, 21, 22]. Their applications in partial differential equations were outlined [23, 24, 25, 26, 27]. This paper is devoted to families of quasi-permutable normal operators in octonion Hilbert spaces. Their spectra are studied. Multiparameter semigroups of such operators are considered. A non-associative analog of Stone's theorem is proved.

Notations and definitions of papers [18, 19, 20, 21, 22] are used below. The main results of this article are obtained for the first time.

## 2 Quasi-permutability of normal operators

**1. Definitions.** If  ${}_jA$  is a set of  $\mathbf{R}$  homogeneous  $\mathcal{A}_v$  additive operators with  $\mathcal{A}_v$  vector domains  $\mathcal{D}({}_jA)$  dense in a Hilbert space  $X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$ ,  $2 \leq v$ ,  $j \in \Lambda$ ,  $\Lambda$  is a set, then we denote by  $alg_{\mathcal{A}_v}({}_jA : j \in \Lambda)$  a family of all operators  $B$  with  $\mathcal{A}_v$  vector domains in  $X$  obtained from  $({}_jA : j \in \Lambda)$  by a finite number of operator addition, operator multiplication and left and right multiplication of operators on Cayley-Dickson numbers  $b \in \mathcal{A}_v$  or on  $bI$ , where  $I$  denotes the unit operator on  $X$ .

Let  ${}_1A$  and  ${}_2A$  be two normal operators in a Hilbert space  $X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$ ,  $2 \leq v$ . Suppose that  ${}_1A$  and  ${}_2A$  are affiliated with a quasi-commutative von Neumann algebra  $\mathbf{A}$  over  $\mathcal{A}_v$  with  $2 \leq v \leq 3$ . Let  ${}_1E$  and  ${}_2E$  be their  $\mathcal{A}_v$  graded projection valued measures defined on the Borel  $\sigma$ -algebra of subsets in  $\mathcal{A}_v$  (see also §2 and §§I.2.58 and I.2.73 in [28]). In this section the simplified notation  $E$  instead of  $\hat{\mathbf{E}}$  will be used.

We shall say that two normal operators  ${}_1A$  and  ${}_2A$  quasi-permute if

$$(1) \quad {}_1E(\delta_1) {}_2E(\delta_2) = {}_2E(\delta_2) {}_1E(\delta_1)$$

for each Borel subsets  $\delta_1$  and  $\delta_2$  in  $\mathcal{A}_v$ .

Operators  $A$ ,  ${}_1A$  and  ${}_2A$  are said to have property  $P$  if they satisfy the following four conditions ( $P1 - P4$ ):

( $P1$ ) they are normal,

(P2) they are affiliated with a von Neumann algebra  $\mathbf{A}$  over either the quaternion skew field or the octonion algebra  $\mathcal{A}_v$  with  $2 \leq v \leq 3$  and

$$(P3) \quad A = {}_1A {}_2A \text{ and}$$

(P4) the family  $\text{alg}_{\mathcal{A}_v}(I, A, A^*, {}_1A, {}_1A^*, {}_2A, {}_2A^*) =: \mathbf{Q}(A, {}_1A, {}_2A) =: \mathbf{Q}$  over  $\mathcal{A}_v$  generated by these three operators is quasi-commutative, that is a von Neumann algebra

$$\text{cl}[\text{alg}_{\mathcal{A}_v}(I, AE(\delta), A^*E(\delta), {}_1A {}_1E(\delta_1), {}_1A^* {}_1E(\delta_1), {}_2A {}_2E(\delta_2), {}_2A^* {}_2E(\delta_2))] \subset L_q(X)$$

contained in  $L_q(X)$  is quasi-commutative for each bounded Borel subsets  $\delta, \delta_1, \delta_2 \in \mathcal{B}(\mathcal{A}_v)$ , where  $2 \leq v \leq 3$ .

It is possible to consider a common domain  $\mathcal{D}^\infty(\mathbf{Q}) := \cap_{T \in \mathbf{Q}} \mathcal{D}^\infty(T)$  for a family of operators  $\mathbf{Q}$ , where  $\mathcal{D}^\infty(T) := \cap_{n=1}^\infty \mathcal{D}(T^n)$ . Then the family  $\mathbf{Q}$  on  $\mathcal{D}^\infty(\mathbf{Q})$  can be considered as an  $\mathcal{A}_v$  vector space. Take the decomposition  $\mathbf{Q} = \mathbf{Q}_0 i_0 \oplus \mathbf{Q}_1 i_1 \oplus \dots \oplus \mathbf{Q}_{2^v-1} i_{2^v-1}$  of this  $\mathcal{A}_v$  vector space with pairwise isomorphic real vector spaces  $\mathbf{Q}_0, \mathbf{Q}_1, \dots, \mathbf{Q}_{2^v-1}$ . Then as in §2.5 [29] for each operator  $B \in \mathbf{Q}$  we put

$$(2) \quad B = \sum_j {}^j B \text{ with } {}^j B = \hat{\pi}^j(B) \in \mathbf{Q}_j i_j$$

for each  $j$ , where  $\hat{\pi}^j : \mathbf{Q} \rightarrow \mathbf{Q}_j i_j$  is the natural  $\mathbf{R}$  linear projection, real linear spaces  $\mathbf{Q}_j i_j$  and  $i_j \mathbf{Q}_j$  are considered as isomorphic, so that

$$(3) \quad \sum_{k=0}^{2^v-1} {}^k \hat{T} = T.$$

If  $E$  is an  $\mathcal{A}_v$  graded projection valued measure on the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathcal{A}_v)$  for a normal operator  $T \in \mathbf{Q}$ , for uniformity of this notation we put also

$$(4) \quad {}^k \hat{E}(dz).ty = \hat{\pi}^k E(dz).ty$$

for every vector  $y \in X$  and  $t = t_0 i_0 + \dots + t_{2^v-1} i_{2^v-1} \in \mathcal{A}_v$ , where  $z \in \mathcal{A}_v$ ,  $t_0, \dots, t_{2^v-1} \in \mathbf{R}$ ,  $E(dz).ty = E(dz).(ty)$ .

**2. Lemma.** *Let operators  $A, B$  and  $D$  have property  $P$  and let  $F$  be an  $\mathcal{A}_v$  graded projection operator which quasi-permutes with  $A$  so that  $\mathcal{R}(F) \subset \mathcal{D}(A)$ , where  $\mathcal{D}(A) = \text{Domain}(A)$ ,  $\mathcal{R}(A) = \text{Range}(A)$ . Suppose that  $G, H$  and  $J$  are the restrictions of  $A, FB$  and  $FD$  to  $\mathcal{R}(F)$  respectively.*

Then  $G$ ,  $H$  and  $J$  are bounded operators so that  $H$  and  $J$  quasi-permute with  $G$ . Moreover,  $H^*$  and  $J^*$  are the restrictions to  $\mathcal{R}(F)$  of  $B^*F^*$  and  $C^*F^*$  respectively, where

$$(1) \quad {}^j(\hat{B}^*)^k(\hat{F}^*) = (-1)^{\kappa(j,k)+\eta(k)} {}^k\hat{F}^j({}^j\hat{B}^*) \text{ and} \\ (2) \quad {}^j(\hat{D}^*)^k(\hat{F}^*) = (-1)^{\kappa(j,k)+\eta(k)} {}^k\hat{F}^j({}^j\hat{D}^*)$$

for each  $j, k$ , with  $\kappa(j, k) = 0$  for  $j = k$  or  $j = 0$  or  $k = 0$ ,  $\kappa(j, k) = 1$  for  $j \neq k \geq 1$ ,  $\eta(0) = 0$ ,  $\eta(k) = 1$  for each  $k \geq 1$ .

**Proof.** Note 2.5 and Theorems 2.29, 2.44 and Proposition 2.32 in [29] and Definitions 1 imply that in components the following formulas are satisfied:

$$(3) \quad {}^j_1\hat{E}(\delta_1) {}^k_2\hat{E}(\delta_2) = (-1)^{\kappa(j,k)} {}^k_2\hat{E}(\delta_2) {}^j_1\hat{E}(\delta_1) = {}^j_2\hat{E}(\delta_2) {}^k_1\hat{E}(\delta_1)$$

for each  $j, k = 0, 1, 2, \dots$ , where  $\kappa(j, k) = 0$  for  $j = k$  or  $j = 0$  or  $k = 0$ ,  $\kappa(j, k) = 1$  for  $j \neq k \geq 1$ ,

where  $\theta_k^j(x_j)$  is denoted by  $x_j$  for short,  $\theta_k^j : X_j \rightarrow X_k$  is an  $\mathbf{R}$ -linear topological isomorphism of real normed spaces (see §§I.2.1 and I.2.73 in [28]). Suppose that  $x, y \in \mathcal{R}(F)$ , hence  $x, y \in \mathcal{D}(B) = \mathcal{D}(B^*)$ , since  $\mathcal{R}(F) \subset \mathcal{D}(A) \subset \mathcal{D}(B)$ .

Therefore

$$(4) \quad \langle FBx; y \rangle = \langle Bx; y \rangle = \langle x; B^*y \rangle = \langle x; B^*y \rangle = \langle x; FB^*y \rangle \text{ and} \\ (5) \quad \langle {}^j\hat{F}^k\hat{B}x_k; y_j \rangle = \langle {}^k\hat{B}x_k; y_j \rangle i_j^* = (-1)^{\kappa(j,k)+\eta(k)} \langle x_k; {}^j\hat{F}^k({}^j\hat{B}^*)y_j \rangle.$$

If  $L = FB^*|_{\mathcal{R}(F)}$ , then  $H^* = L$  and  $H = L^*$  by Formula (4). The operator  $L^*$  is closed, consequently,  $H$  is closed and  $\mathcal{D}(H) \supset \mathcal{R}(F)$ . In view of the closed graph theorem for  $\mathbf{R}$ -linear operators the operator  $H$  is bounded 1.8.6 [12]. This implies that the operator  $G$  is also bounded, since the operator  $A$  is normal and hence closed so that  $\mathcal{R}(F) \subset \mathcal{D}(A)$ . In view of Theorems 2.27, 2.29 and 2.44 in [29] the operator  $A$  has an  $\mathcal{A}_v$  graded projection valued measure. Take now  $x \in \mathcal{R}(F)$ , hence  $Ax \in \mathcal{R}(F) \subset \mathcal{D}(A) \subset \mathcal{D}(B)$ , since

$${}^j\hat{F}^k\hat{F} = (-1)^{\kappa(j,k)} {}^k\hat{F}^j\hat{F} \text{ and } \mathcal{D}(F) = \mathcal{D}(F)_{0i_0} \oplus \dots \oplus \mathcal{D}(F)_{mi_m} \oplus \dots$$

for each  $j, k$  and

$$A = \int_{\mathcal{A}_v} F(dt).t$$

so that  ${}^j\hat{F}^k\hat{A} \subseteq (-1)^{\kappa(j,k)} {}^k\hat{A}^j\hat{F}$  for each  $j, k$ . Symmetric proof is for  $A$  and  $C$  instead of  $A$  and  $B$ . The operators  $B^*B$  and  $C^*C$  belong to the family  $\text{alg}_{\mathcal{A}_v}(I, A, A^*, B, B^*, C, C^*)$ .

In view of Theorem I.3.23 [28] the spectra of  $B^*B = \int_{-\infty}^{\infty} B^*B F(dt).t^2$  and  $D^*D = \int_{-\infty}^{\infty} D^*D F(dt).t^2$  are real so that  $B^*B F$  and  $D^*D F$  are  $\mathcal{A}_v$  graded projection valued measures for  $B^*B$  and  $D^*D$  respectively on  $\mathcal{B}(\mathbf{R}) \subset \mathcal{B}(\mathcal{A}_r)$ .

Then from Formulas (2, 4) and 1(1, P1 – P4) we deduce that

$$\begin{aligned} (7) \quad & ({}^j\hat{F} \ {}^k\hat{B}) \ {}^s\hat{A}x_s = ({}^j\hat{F} \ {}^k\hat{B}) \sum_{p,q: i_p i_q = i_s} [{}^p\hat{D} \ {}^q\hat{B} + (-1)^{\kappa(p,q)} \ {}^q\hat{D} \ {}^p\hat{B}] \\ & = \sum_{p,q: i_p i_q = i_s} [({}^j\hat{F} \ {}^k\hat{B}) ({}^p\hat{D} \ {}^q\hat{B}) + (-1)^{\kappa(p,q)} ({}^j\hat{F} \ {}^k\hat{B}) ({}^q\hat{D} \ {}^p\hat{B})] \\ & = (-1)^{\kappa(s,l)} \ {}^s\hat{A}({}^j\hat{F} \ {}^k\hat{B}x_s), \end{aligned}$$

since the set theoretic composition of operators is associative:  $(FB)(DB) = F((BD)B)$ , where  $l$  is such that  $i_j i_k \in \mathbf{R}i_l$ . Thus  $H$  and analogously  $J$  quasi-permute with  $G$ , since the family  $\text{alg}_{\mathcal{A}_v}(I, A, A^*, {}_1A, {}_1A^*, {}_2A, {}_2A)$  is quasi-commutative. From Formulas (5, 6) we infer Equalities (1, 2).

**3. Notation.** Suppose that  $a, b \in \mathcal{A}_r$ . If  $b_j \geq a_j$  for each  $j = 0, 1, 2, \dots, 2^r - 1$ , this fact will be denoted by  $b \succeq a$ . Then  $\mathcal{I}_{a,b} := \{z \in \mathcal{A}_r : b \succeq z \succeq a\}$ .

**4. Lemma.** Let operators  $A, B$  and  $D$  have property  $P$  and let  $F$  be an  $\mathcal{A}_v$  graded projection valued measure for  $A$ , let also  $b \succeq a \in \mathcal{A}_v$ . Then  $\mathcal{R}(F(\mathcal{I}_{a,b})) =: Y$  reduces both  $B$  and  $D$  and these operators restricted to  $Y$  are bounded and normal and they quasi-permute with the restriction  $A|_Y$ .

**Proof.** Consider the pair of operators  $A$  and  $B$ . Put  ${}_nF := F|_{\mathcal{I}_{-b(n), b(n)}}$  and  ${}_nV = \mathcal{R}(F(\mathcal{I}_{-b(n), b(n)}))$  with  $b(n)_j = ni_j$  for every  $n \in \mathbf{N}$  and each  $j = 0, 1, 2, \dots, 2^v - 1$ . Then  ${}_nV \subset {}_{n+1}V$  for each  $n$ . Therefore, an  $\mathcal{A}_v$  vector subspace  $\bigcup_n {}_nV =: V$  is dense in the Hilbert space  $X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$ , consequently,  $\lim_n {}_nF = I$  in the strong operator topology. Each operator  ${}_nA := A|_{{}_nV}$  is bounded and normal and has the  $\mathcal{A}_v$  graded projection valued measure on the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathcal{A}_v)$  of all Borel subsets in  $\mathcal{A}_v$  so that  ${}_nF = F|_{{}_nV}$  for each natural number  $n$ . We consider the restriction  ${}_nG := {}_nF B|_{{}_nV}$ . It is known from Lemma 2, that each operator  ${}_nG$  is bounded and quasi-permutes with  ${}_nB$  so that

$$(1) \quad {}_nG {}^j\hat{F} \ {}^k_{nB}\hat{F} = (-1)^{\kappa(j,k)} \ {}^k_{nB}\hat{F} \ {}^j_{nG}\hat{F}$$

for each  $j, k$ , consequently,

$$(2) \quad {}^s_n\hat{F}(\delta)({}^j_{nB}\hat{F}(\delta_1) \ {}^k_n\hat{F}(\delta_2)x) = (-1)^{\kappa(j,k)} \ {}^s_n\hat{F}(\delta)({}^k_n\hat{F}(\delta_2) \ {}^j_{nB}\hat{F}(\delta_1)x)$$

for each  $x \in {}_nV_0$  and  $\delta, \delta_1, \delta_2 \in \mathcal{B}(\mathcal{A}_v)$ , where  ${}_nG$  and  ${}_nB$  denote  $\mathcal{A}_v$  graded projection valued measures for the operators  ${}_nG$  and  ${}_nB$  correspondingly.

Let now  $y \in \mathcal{D}(A)_0$  and  $\delta \in \mathcal{B}(\mathcal{A}_v)$  be fixed, hence

$$(3) \quad \lim_n {}^s\hat{F}(\delta)({}^k\hat{F}(\delta_2){}^j\hat{F}(\delta_1)x) = \lim_n {}^s\hat{F}(\delta)({}^k\hat{F}(\delta_2){}^j\hat{F}(\delta_1)x) \\ = \pm(-1)^{\psi(s,k,j)} {}^l\hat{F}(\delta \cap \delta_2){}^j\hat{F}(\delta_1)x,$$

where  $i_si_k = \pm i_l$ ,  $\psi(s, j, k) \in \{0, 1\}$  is an integer so that  $i_s(i_ji_k) = (-1)^{\psi(s,j,k)}(i_si_j)i_k$ .

If a vector  $x \in \bigcup_n {}_nV_0$  is given, then there exists a natural number  $m$  such that

$$(4) \quad {}^s\hat{F}(\delta)({}^j\hat{F}(\delta_1){}^k\hat{F}(\delta_2)x) = {}^s\hat{F}(\delta)({}^j\hat{F}(\delta_1){}^k\hat{F}(\delta_2)x)$$

for each  $n > m$ , consequently,

$$(5) \quad \lim_n {}^s\hat{F}(\delta)({}^j\hat{F}(\delta_1){}^k\hat{F}(\delta_2)x) = {}^s\hat{F}(\delta)({}^j\hat{F}(\delta_1){}^k\hat{F}(\delta_2)x),$$

where  $F(\mathcal{A}_v) = I$ ,  $I$  denotes the unit operator. From Formulas (2 – 5) and the inclusions  $\bigcup_n {}_nV =: V \subset \mathcal{D}(A) \subset \mathcal{D}(B)$  it follows, that

$$(6) \quad {}^j\hat{B}({}^k\hat{F}(\delta)x_si_s) = (-1)^{\xi(j,k,s)} {}^k\hat{F}(\delta)({}^j\hat{B}x_si_s)$$

for each  $x_si_s \in V$  and  $j, k, s = 0, 1, 2, \dots$ , where  $\xi(j, k, s) \in \{0, 1\}$  is such integer number that  $i_j(i_ki_s) = (-1)^{\xi(j,k,s)}i_k(i_ji_s)$ . From the formula  $i_j(i_ki_s) + i_k(i_ji_s) = 2i_s \text{Re}(i_ji_k)$  we get  $(-1)^{\xi(j,k,s)} = (-1)^{\kappa(j,k)}$  for each  $j, k$  and  $s$ , since an algebra  $\text{alg}_{\mathbf{R}}(i_j, i_k, i_s)$  over  $\mathbf{R}$  generated by  $i_j, i_k$  and  $i_s$  has an embedding into the octonion algebra which is alternative [1] (see also Formulas 4.2.4(7, 8) in [22]). Thus  $BV \subset V$  and  ${}_BFV \subset V$ . Then

$$(7) \quad {}^j\hat{F}(\delta_1){}^k\hat{F}(\delta) = (-1)^{\kappa(j,k)} {}^k\hat{F}(\delta){}^j\hat{F}(\delta_1),$$

that is  ${}_nH^*$  quasi-permutes with  ${}_nF$ .

In view of Lemma 2 we have  ${}_nH^* = B^*|_{{}_nV}{}_nF(\mathcal{I}_{-b(n),b(n)})$  and from the proof above we get

$$(8) \quad {}^j\hat{F}({}^k\hat{F}^*(x_si_s)) = (-1)^{\xi(j,k,s)} {}^k\hat{F}^*({}^j\hat{F}x_si_s) \\ = (-1)^{\xi(j,k,s)} {}^k\hat{F}^*({}^j\hat{I}x_si_s),$$

consequently,  $B({}_nV) \subset {}_nV$  and  ${}_BF({}_nV) \subset {}_nV$ . Consider decomposition  $x = y + z$  with  $y \in {}_nV$  and  $z \in {}_nV^\perp$ , then  $x \in \mathcal{D}(B)$  is equivalent to

$z \in \mathcal{D}(B)$ . The latter inclusion implies  $z \in \mathcal{D}(B) \cap {}_nV$ , if additionally  $x \in {}_nV$ , then we get  $\langle B^*y; z \rangle = \langle y; Dz \rangle = 0$ , consequently,  $Bz \in {}_nV$  and this together with (7) leads to the inclusion  ${}_nFB \subset B {}_nF$ , that is  ${}_n\hat{F} {}^k\hat{B} \subset (-1)^{\kappa(j,k)} {}^k\hat{B} {}_n\hat{F}$ . For any  $\mathbf{R}$ -linear spaces a sign in an inclusion does not play any role. Thus  ${}_nV$  reduces  $B$  and  ${}_nG$  into a normal operator  ${}_nQ = B|_{{}_nV}$ .

Suppose that  ${}_nG F$  is the canonical  $\mathcal{A}_v$  graded projection valued measure for  ${}_nG$  and  ${}_BF$  is the canonical  $\mathcal{A}_v$  graded projection valued measure for  $B$ , hence  ${}_BF|_{{}_nV} = {}_nG F$  for each  $n \in \mathbf{N}$ . If  $x \in \bigcup_n {}_nV$ , there exists a natural number  $m$  so that

$$\begin{aligned} (9) \quad {}^j\hat{E}(\delta_1)({}^k\hat{F}(\delta_2)x_si_s) &= {}^j_{{}_nG}\hat{F}(\delta_1)({}^k_{{}_nG}\hat{F}(\delta_2)x_si_s) \\ &= (-1)^{\xi(j,k,s)} {}^k_{{}_nG}\hat{F}(\delta_2)({}^j_{{}_nG}\hat{F}(\delta_1)x_si_s) \end{aligned}$$

for each Borel subsets  $\delta_1$  and  $\delta_2$  in  $\mathcal{A}_r$ , since the restriction of  $A$  to  ${}_nV$  and  ${}_nG$  quasi-permute for all  $n$  in accordance with Lemma 2. On the other hand, the  $\mathcal{A}_v$  vector space  $V$  is dense in  $X$ , consequently,  ${}_BF$  and  $F$  quasi-permute:

$$(10) \quad {}^j_B\hat{F}(\delta_1) {}^k\hat{F}(\delta_2)x_0 = (-1)^{\kappa(j,k)} {}^k\hat{F}(\delta_2) {}^j_B\hat{F}(\delta_1)x_0$$

for each  $j, k = 0, 1, 2, \dots$  and  $x_0 \in X_0$ .

If now  $F$  is an  $\mathcal{A}_v$  graded projection valued measure described in this lemma, then Formula (10) implies

$$(11) \quad {}^k\hat{F} {}^j\hat{B} \subseteq (-1)^{\kappa(j,k)} {}^j\hat{B} {}^k\hat{F}$$

for each  $j, k = 0, 1, 2, \dots, 2^v - 1$ , consequently,  $\mathcal{R}(F)$  reduces  $B$  and  $B|_{\mathcal{R}(F)}$  is a normal operator with  $\mathcal{R}(F) \subset \mathcal{D}(B)$ , since  $\mathcal{R}(F) \subset \mathcal{D}(A) \subset \mathcal{D}(B)$ . This restriction  $B|_{\mathcal{R}(F)}$  is bounded by the closed graph theorem 1.8.6 [12]. Moreover, the restrictions of  $A$  and  $B$  to  $\mathcal{R}(F)$  quasi-permute. Analogous proof is valid for the pair  $A$  and  $C$  instead of  $A$  and  $B$ .

**5. Theorem.** *If operators  $A$ ,  $B$  and  $D$  satisfy property  $P$ , then  $B$  and  $D$  quasi-permute so that*

$$(1) \quad {}^j\hat{B} {}^k\hat{D} = (-1)^{\kappa(j,k)} {}^k\hat{D} {}^j\hat{B}$$

for each  $j, k$ . Moreover,

$$(2) \quad {}^l\hat{A} = \sum_{j,k; i_j i_k = i_l} ({}^j\hat{B} {}^k\hat{D} + (-1)^{\kappa(j,k)} {}^k\hat{B} {}^j\hat{D})$$

for each  $l$ .

**Proof.** Consider the canonical  $\mathcal{A}_v$  graded projection valued measure  $E$  for a normal operator  $A$  (see Definition 1). Then we put  ${}_nF := E(\mathcal{I}_{a,b})$  with  $a_j = -ni_j$  and  $b_j = ni_j$  for each  $j$ . From Theorems 2.27, 2.29 and 2.44 in [29] and §4 above we know that

$$(3) \quad Ax = \int_{\mathcal{A}_v} d_A E(t).tx \quad \forall x \in \mathcal{D}(A) \text{ and}$$

$$(4) \quad Bx = \int_{\mathcal{A}_r} d_B E(t).tx \quad \forall x \in \mathcal{D}(B) \text{ and}$$

$$(5) \quad Dx = \int_{\mathcal{A}_v} d_D E(t).tx \quad \forall x \in \mathcal{D}(D),$$

where  ${}_A E$ ,  ${}_B E$  and  ${}_D E$  denote  $\mathcal{A}_v$  graded projection valued measures for  $A$ ,  $B$  and  $D$  respectively. Then the condition  $A = BD$  gives

$$(6) \quad Ax = \int_{\mathcal{A}_v} d_B E(t).t \int_{\mathcal{A}_v} d_D E(u).ux.$$

To operators  $A$ ,  $B$  and  $D$  normal functions  $h_A$ ,  $h_B$  and  $h_D$  correspond so that  $h_A = h_B h_D$ . On the other hand, to the operators  $A^*A$  and  $B^*B$  and  $D^*D$  non-negative self-adjoint functions  $|h_A|^2$ ,  $|h_B|^2$  and  $|h_D|^2$  correspond (see Proposition 2.32 in [29]). These operators  $A$  and  $B$  and  $D$  are normal so that they satisfy the identities  $A^*A = D^*B^*BD = D^*BB^*D = AA^* = BDD^*B^* = BD^*DB^*$  and  $B^*B = B^*B$  and  $D^*D = DD^*$ .

In view of Theorems 2.29, 2.44 and Proposition 2.32 and Remark 2.43 in [29] to the  $\mathcal{A}_v$  graded projection operator  ${}_A E(\delta)$  a homomorphism  $\phi$  a (real) characteristic function  $\phi({}_A E(\delta)) = \chi_u$  of a subset  $u \subset \Lambda$  counterpose so that  $\chi_u = \omega(\chi_\delta)$ . Therefore, Theorem 2.23 and Lemma 2.21 in [29], Formulas (3 – 6) and Conditions (P1 – P4) imply that their projection operators satisfy the equality

$$(7) \quad {}_B E(\delta_1) {}_D E(\delta_2) = {}_D E(\delta_2) {}_B E(\delta_1)$$

for each Borel subsets  $\delta_1$  and  $\delta_2$  in  $\mathcal{A}_v$ . In view of Lemma 4  $\mathcal{R}({}_nF)$  reduces  $B$  and  $D$  and the restrictions of these operators to  $\mathcal{R}({}_nF)$  are bounded normal operators. On the other hand,  $\bigcup_{n=1}^\infty \mathcal{R}({}_nF)$  is dense in the Hilbert space



$X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$ . Therefore, we infer from Formulas (3 – 7), that  ${}^jB$  and  ${}^kD$  satisfy Formulas (1, 2) for each  $j, k$ , since

$$(8) \quad {}^j_B \hat{E}(\delta_1) {}^k_D \hat{E}(\delta_2) = (-1)^{\kappa(j,k)} {}^k_D \hat{E}(\delta_2) {}^j_B \hat{E}(\delta_1)$$

for every Borel subsets  $\delta_1$  and  $\delta_2$  in  $\mathcal{A}_v$  and for each  $j, k$ .

**6. Corollary.** *Suppose that operators  $A$ ,  $B$  and  $D$  are self-adjoint and satisfy property (P). Then  $BD = DB$ .*

**Proof.** This follows immediately from Theorems 2.27, 2.29 and 2.44 in [29] and Formulas 5(1 – 3), since spectra of self-adjoint operators are contained in the real field  $\mathbf{R}$  and the latter is the center of the Cayley-Dickson algebra  $\mathcal{A}_v$  so that  $t = t_0 \in \mathbf{R}$  in Formulas 5(1, 2), that is  $j = k = 0$  only.

**7. Lemma.** *Let operators  $B$ ,  $D$  and  $A$  have property P, let also  $B = T_B U_B$ ,  $D = T_D U_D$  and  $A = TU$  be their canonical decompositions with positive self-adjoint operators  $T_B$ ,  $T_D$  and  $T$  and unitary operators  $U_B$ ,  $U_D$  and  $U$  respectively. Then  $T_B T_D = T_D T_B = T$  and  $U_B U_D = U$  so that  ${}^j U_B {}^k U_D = (-1)^{\kappa(j,k)} {}^j U_B {}^k U_D$  for each  $j, k$ , moreover,  $T_B U_D = U_D T_B$  and  $T_D U_B = U_B T_D$ .*

**Proof.** The decompositions in the conditions of this lemma are particular cases of that of Theorem I.3.37 [28]. Consider the canonical  $\mathcal{A}_v$  graded resolutions of the identity  $E^B$  and  $E^D$  of operators  $B$  and  $D$  respectively. In view of Theorem 5

$${}^j E^B(\delta_1) {}^k E^D(\delta_2) = (-1)^{\kappa(j,k)} {}^k E^D(\delta_2) {}^j E^B(\delta_1)$$

for every Borel subsets  $\delta_1$  and  $\delta_2$  in  $\mathcal{A}_v$  and each  $j, k$ . We put  $F(dw, dz) = E^B(dw)E^D(dz)$ , hence  $F(dw, dz)$  is a  $2^{v+1}$  parameter  $\mathcal{A}_v$  graded resolution of the identity so that  $F_{i_k}(\delta_1, \delta_2)x_k = E^B(\delta_1)(E^D)_{i_k}(\delta_2)$  for each vector  $x_k \in X_k$  and every  $k$  and we put

$$G := \int_{\mathcal{A}_v^2} dF(w, z).wz,$$

where  $dF(w, z)$  is another notation of  $F(dw, dz)$ ,  $w, z \in \mathcal{A}_v$  (see also §I.2.58 [28]). This operator  $G$  is normal, since the quaternion skew field is associative

and the octonion algebra is alternative and  $(wz)(wz)^* = |wz|^2 = |w|^2|z|^2$  for each  $w, z \in \mathcal{A}_v$  with  $2 \leq v \leq 3$ . Then we get

$$B = \int_{\mathcal{A}_v^2} dF(w, z).w = \int_{\mathcal{A}_v} dE^B(w).w \text{ and}$$

$$D = \int_{\mathcal{A}_v^2} dF(w, z).z = \int_{\mathcal{A}_v} dE^D(z).z, \text{ consequently,}$$

$A = BD$  and  ${}^j B {}^k D = (-1)^{\kappa(j,k)} {}^k D {}^j B$  for each  $j, k$ , and hence

$$\sum_{j,k: i_j i_k = i_l} [{}^j B {}^k D + (-1)^{\kappa(j,k)} {}^k D {}^j B] \subseteq {}^l G$$

for every  $l$ . Therefore,  $A = G$ , since a normal operator is maximal.

Then one can consider the function  $u(w, z) := wz/|wz|$  for  $wz \neq 0$ , while  $u(w, z) = 1$  if  $wz = 0$ , where  $w, z \in \mathcal{A}_v$ . The operator

$$U := \int_{\mathcal{A}_v^2} dF(w, z).u(w, z)$$

is unitary, since  $|u(w, z)| = 1$  for each  $w$  and  $z$ , the operator

$$T := \int_{\mathcal{A}_v^2} dF(w, z).|wz|$$

is positive and self-adjoint, since

$$\langle xT; x \rangle := \int_{\mathcal{A}_v^2} \langle x dF(w, z).|wz|; x \rangle \geq 0$$

for each  $x \in \mathcal{D}(T)$  (see Proposition 2.35 [29]). On the other hand,  $u(w, z)|wz| = |wz|u(w, z) = wz$ , since the algebra  $\mathcal{A}_v$  is alternative for  $v \leq 3$ , hence  $TU = UT = G = A$  by Theorem 2.44 [29]. Moreover, we deduce from Theorem 2.44 [29] that the operators

$$U_B := \int_{\mathcal{A}_v^2} dF(w, z).u(w) \text{ and}$$

$$U_D := \int_{\mathcal{A}_v^2} dF(w, z).u(z)$$

are unitary and the the operators

$$T_B := \int_{\mathcal{A}_v^2} dF(w, z).|w| \text{ and}$$

$$T_D := \int_{\mathcal{A}_v^2} dF(w, z).|z|$$

are positive and self-adjoint, where  $u(w) := w/|w|$  if  $w \neq 0$ , also  $u(w) = 1$  if  $w = 0$ . Since  $|w||z| = |wz|$  for each  $w$  and  $z \in \mathcal{A}_v$  with  $v \leq 3$ , the inclusion follows

$$T_B T_D \subseteq \int_{\mathcal{A}_v^2} dF(w, z) \cdot |wz| = T.$$

The functions  $u(w)$  and  $u(z)$  are bounded and  $u(w)u(z) = u(z)u(w) = u(w, z)$  on  $\mathcal{A}_v^2$ , consequently,

$$U_B U_D = \int_{\mathcal{A}_v^2} dF(w, z) \cdot u(w, z) = U \text{ so that}$$

${}^j U_B {}^k U_D = (-1)^{\kappa(j,k)} {}^k U_B {}^j U_D$  for each  $j, k$ . This implies that  $A = UT = U_B U_D T = (U_B T_B)(U_D T_D) = (U_B T_B)(T_D U_D)$ , consequently,  $U_D T U_D^* = T_B T_D$ . This means that the operators  $T$  and  $T_B T_D$  are unitarily equivalent, hence the operator product  $T_B T_D$  is self-adjoint. A self-adjoint operator is maximal, consequently,  $T = T_B T_D$  and similarly  $T = T_D T_B$ . The real field  $\mathbf{R}$  is the center of the Cayley-Dickson algebra  $\mathcal{A}_v$  for each  $v \geq 2$ , the real and complex fields are commutative, hence

$$T_B U_D = \int_{\mathcal{A}_v^2} dF(w, z) \cdot (|w|u(z)) = U_D T_B \text{ and}$$

$$T_D U_B = \int_{\mathcal{A}_v^2} dF(w, z) \cdot (|z|u(w)) = U_B T_D.$$

**8. Notation.** Let  $\Omega$  denote the set of all  $n$ -tuples  $x = (x_1, \dots, x_m, x_{m+1}, \dots, x_n)$  such that  $x_1, \dots, x_m$  are non-negative integers, while  $x_{m+1}, \dots, x_n$  are non-negative real numbers with  $\sum_{j=1}^n x_j > 0$ . Relative to the addition  $x + y = (x_1 + y_1, \dots, x_n + y_n)$  this set  $\Omega$  forms a semi-group.

**9. Theorem.** Suppose that  $\{B^x : x \in \Omega\}$  is a weakly continuous semi-group of normal operators, that is satisfying the following conditions:

(1)  $B^x$  is a normal operator acting on a Hilbert space  $X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$  for each element  $x \in \Omega$ ;

(2)  $B^x B^y = B^{x+y}$  for each  $x, y \in \Omega$ ;

(3) the  $\mathcal{A}_v$  valued scalar product  $\langle B^x f; g \rangle$  is continuous in  $x \in \Omega$  for each marked  $f, g \in \mathcal{D} := \bigcap_{x \in \Omega} \mathcal{D}(B^x)$ ;

(4) a family  $\text{alg}_{\mathcal{A}_v}\{I, B^x, (B^x)^* : x \in \Omega\}$  is over the algebra  $\mathcal{A}_v$  with  $2 \leq v \leq 3$ . Then a unique  $2n$ -parameter  $\mathcal{A}_v$  graded resolution  $\{(a_1, \dots, a_n; b_1, \dots, b_n) \hat{F} : a, b \in$

$\Omega\}$  of the identity exists so that  ${}_{(a,b)}\hat{F} = 0$  if a negative coordinate  $a_k < 0$  exists for some  $k = 1, \dots, n$ , moreover,

$$(5) \quad B^x = \int_{\mathbf{R}^{2n}} d {}_{(a,b)}\hat{F} \cdot \{a^x \exp[x_1 M_1(b_1)b_1] \dots \exp[x_n M_n(b_n)b_n]\},$$

where

$$a^x = \prod_{k=1}^n a_k^{x_k},$$

$M_s : \mathbf{R}^n \rightarrow \mathcal{S}_v := \{z \in \mathcal{A}_v : |z| = 1, \operatorname{Re}(z) = 0\}$  is a Borel function for each  $s$ ,  $a = (a_1, \dots, a_n)$ .

**Proof.** In view of Lemma 5 each operator  $B^x$  has the decomposition  $B^x = T^x U^x = U^x T^x$  with a positive self-adjoint operator  $T^x$  and a unitary operator  $U^x$ . Since  $\{B^x : x \in \Omega\}$  is a semi-group, the relations  $T^x T^y = T^{x+y}$  and  $U^x U^y = U^{x+y}$  are valid for each elements  $x, y \in \Omega$ . That is,  $\{T^x : x \in \Omega\}$  and  $\{U^x : x \in \Omega\}$  are semi-groups of positive self-adjoint operators and unitary operators correspondingly.

If  $y^s = (0, \dots, y_{m+1}^s, \dots, y_n^s) \in \Omega$  are elements of the semi-group  $\Omega$  such that  $y^1 = \frac{y^2 + y^3}{2}$ ,  $s = 1, 2, 3$ ,  $f$  is a vector in a domain  $\mathcal{D}$ , then

$$\begin{aligned} \|B^{y^1} f\|^2 &= \langle B^{y^1} f, B^{y^1} f \rangle = \langle B^{y^2/2} B^{y^3/2} f, B^{y^2/2} B^{y^3/2} f \rangle \\ &= \langle (B^{y^2/2})^* B^{y^2/2} f, (B^{y^3/2})^* B^{y^3/2} f \rangle \leq \| (B^{y^2/2})^* B^{y^2/2} f \| \| (B^{y^3/2})^* B^{y^3/2} f \| \end{aligned}$$

by Cauchy-Schwartz' inequality I.2.4(1) [28]. On the other hand,

$$\| (B^{y^2/2})^* B^{y^2/2} f \|^2 = \langle (B^{y^2/2})^* B^{y^2/2} f, (B^{y^2/2})^* B^{y^2/2} f \rangle = \langle B^{y^2} f, B^{y^2} f \rangle = \|B^{y^2} f\|^2,$$

since the semi-group  $\{B^x : x \in \Omega\}$  is commutative and an operator  $B^x$  is normal for each  $x \in \Omega$ . Thus the inequality

$$\|B^{y^1} f\| \leq \|B^{y^2} f\| \|B^{y^3} f\|$$

follows. This implies that the function  $q(y) := \|B^y f\|$  is convex and bounded in the variable  $y_p$  in any bounded segment  $[\alpha, \beta] \subset (0, \infty)$ , when other variables  $y_q$  with  $q \neq p$  are zero,  $p = m+1, \dots, n$ , since the exponential  $e^t$  and the natural logarithmic functions  $\ln(t)$  are convex and bounded on each segment  $[\gamma, \delta] \subset (0, \infty)$  and  $\ln q(y^1) \leq \ln q(y^2) + \ln q(y^3)$ .

Evidently, a commutative group  $\hat{\Omega}$  exists for the semi-group  $\Omega$  such that  $\Omega \subset \hat{\Omega} \subset \mathbf{R}^n$  and the function  $q(y)$  can be extended on  $\hat{\Omega}$  so that  $q(0) = \|f\|$

and  $q(-y) = q(y)$  for  $y \in \Omega$ . If  $q$  is continuous on  $\Omega$ , its extension on  $\hat{\Omega}$  can be chosen continuous, since  $\hat{\Omega}$  is a completely regular topological space, i.e.  $T_1$  and  $T_{3.5}$  (see [6]).

If  $\Omega$  is a group the function  $q(y)$  is positive definite, that is by the definition for each  $\lambda_1, \dots, \lambda_k \in \mathbf{R} \oplus \mathbf{R}\mathbf{i} =: \mathbf{C}_i$  and  $y^1, \dots, y^k \in \Omega$  the inequality

$$\sum_{j,l} \lambda_j \bar{\lambda}_l q(y^j - y^l) \geq 0$$

is valid, but this inequality follows from the formula

$$\sum_{j,l} \lambda_j \bar{\lambda}_l q(y^j - y^l) = \left\| \sum_j \lambda_j B^{y^j} f \right\|^2$$

and since  $\|x\| \geq 0$  for each  $x \in X$ .

Particularly, for elements  $x^k := (0, \dots, x_k, 0, \dots, 0)$  in the semi-group  $\Omega$  the mapping  $\langle T^{x^k} f; f \rangle$  is continuous in  $x^k$  for each marked vector  $f \in \mathcal{D}$ . Indeed, for  $k = 1, \dots, m$  this is evident, since  $x^k \in \mathbf{N}$  takes values in the discrete space in this case. If  $k = m+1, \dots, n$  one can use the formula  $\langle T^{x^k} f; f \rangle = \langle B^{x^k/2} f, B^{x^k/2} f \rangle = \|B^{x^k/2} f\|^2$  which implies that  $\langle T^{x^k} f; f \rangle$  is a bounded convex function of  $x^k$  in every finite interval  $[\alpha, \beta] \subset (0, \infty)$ , when  $f \in \mathcal{D}$  is a marked vector (see Theorem 2.29 and Formula 2.44(5) [29]).

Denote by  ${}_{s,t_s}E$  the canonical  $\mathcal{A}_v$  graded resolution of the identity for  $T^{e_s}$ , where  $e_s = (0, \dots, 0, 1, 0, \dots)$  denotes the basic vector with coordinate 1 at  $s$ -th place and zeros otherwise,  $t_s \in \mathbf{R}$ . By the conditions of this theorem operators  $T^{e_s}$  and  $T^{e_p}$  commute for each  $s, p = 1, \dots, n$ , since

$$(6) \quad T^{e_s} T^{e_p} = T^{e_s + e_p} = T^{e_p} T^{e_s}.$$

Due to Theorem 2.42 [29] the equality

$$(7) \quad {}^j_{s,t_s} E {}^k_{p,t_p} E = (-1)^{\kappa(j,k)} {}^k_{p,t_p} E {}^j_{s,t_s} E$$

is satisfied for each  $j, k$  and every  $s, p$ , with  $t_s, t_p \in \mathbf{R}$ . This implies that

$$(8) \quad (t_1, \dots, t_n) E = {}_{1,t_1} E \dots {}_{n,t_n} E$$

is an  $n$ -parameter  $\mathcal{A}_v$  graded resolution of the identity. Each operator  $T^{e_s}$  is positive, hence  ${}_{s,t_s} E = 0$  for every  $t_s < 0$ , consequently,  $(t_1, \dots, t_n) E = 0$  if  $t_s < 0$  for some  $s = 1, \dots, n$ .

We now consider the operators

$$(9) \quad A^p x := \int_0^\infty \dots \int_0^\infty d {}_{(t_1, \dots, t_n)} E \cdot (t_1^{p_1} \dots t_n^{p_n} x),$$

where  $p = (p_1, \dots, p_n) \in \Omega$ ,  $x \in X$  for which the integral converges. We certainly have

$$\int_0^\infty \dots \int_0^\infty d_{(t_1, \dots, t_n)} E.(t_1^{p_1} \dots t_n^{p_n} x) = \int_0^\infty \dots \int_0^\infty (t_1^{p_1} \dots t_n^{p_n}) d_{(t_1, \dots, t_n)} E.x,$$

since  $t_j^{p_j} \in \mathbf{R}$  for each  $j$  and  $d_{(t_1, \dots, t_n)} E$  is a real linear operator. If  $p_s \in \mathbf{Z}/2$  for each  $s$ , then  $T^p = T^{e_1 p_1} \dots T^{e_n p_n} \subseteq A^p$ , consequently,  $T^p = A^p$ , since a self-adjoint operator is maximal.

Take a partition of the Euclidean space  $\mathbf{R}^n$  into a countable family of bounded parallelepipeds  $J_k = \prod_{j=1}^n [a_j, b_j]$  so that they may intersect only by their boundaries:  $J_k \cap J_l = \partial J_k \cap \partial J_l$  for each  $k \neq l \in \mathbf{N}$ ,  $\bigcup_{k=1}^\infty J_k = \mathbf{R}^n$ . We put  $Y^k := \mathcal{R}(\hat{\mathbf{E}}(J_k))$ , where  $\hat{\mathbf{E}}(\delta)$  is the  $\mathcal{A}_v$  graded spectral measure corresponding to  ${}_t E$ ,  $\delta \in \mathcal{B}(\mathbf{R}^n)$ ,  $t \in \mathbf{R}^n$ . Then the restriction  $B^x|_{Y^k}$  of  $B^x$  to  $Y^k$  is a bounded self-adjoint operator. If  $x, y \in \Omega$  are elements of the semi-group so that  $y_s \geq x_s$  and  $y_s \in \mathbf{Z}/2$  for each  $s = 1, \dots, n$ , then  $\mathcal{D}(T^y) \subseteq \mathcal{D}(T^x)$ , since  $T^y = T^x T^{y-x}$ . Therefore,  $f \in \mathcal{D}(A^y) = \mathcal{D}(T^y) \subseteq \mathcal{D}(T^x)$  for each  $f \in Y^k$ , consequently,  $Y^k \subset \mathcal{D}$  for each natural number  $k \in \mathbf{N}$ .

If  $f \in Y^k \oplus Y^l$  and  $g \in Y^l$ , then

$$\lim_{y \rightarrow x} \langle (T^y - A^y)(f + g); (f + g) \rangle = \langle (T^x - A^x)(f + g); (f + g) \rangle = 0,$$

since  $T^y = A^y$  for each  $y \in (\mathbf{Z}/2)^n \cap \Omega$  and the  $\mathcal{A}_v$  valued scalar products  $\langle T^x f; f \rangle$  and  $\langle A^x f; f \rangle$  are continuous in each component  $x_s$  of  $x$ . In the same manner we get  $\langle (T^x - A^x)f; f \rangle = 0$  and  $\langle (T^x - A^x)g; g \rangle = 0$ , consequently,  $\langle (T^x - A^x)f; g \rangle = 0$ . The  $\mathcal{A}_v$  vector space  $\bigcup_{k=1}^\infty Y^k$  is dense in the Hilbert space  $X$  over the Cayley-Dickson algebra  $\mathcal{A}_v$ , hence  $T^x f^k = (A^x|_{Y^k})f^k = A^x f^k$  for each vector  $f^k \in Y^k$ . This means that each  $Y^k$  reduces the operator  $T^x$  to  $(A^x|_{Y^k})$ , consequently,  $T^x = A^x$ . From this it follows that the  $\mathcal{A}_v$  valued scalar product  $\langle T^x f; g \rangle$  is continuous in  $x \in \Omega$  for each marked vectors  $f \in \mathcal{D}$  and  $g \in X$ .

Consider the sub-semi-group  $\Omega_s := \{x : x = x^s := (0, \dots, 0, x_s, 0, \dots) \in \Omega\}$ , where  $s = 1, \dots, n$ , also we suppose that  ${}_s \hat{\mathbf{E}}(\{0\}) = 0$ , where  ${}_s \hat{\mathbf{E}}(\delta)$  is the  $\mathcal{A}_v$  graded projection valued measure corresponding to  ${}_{s, t_s} E$ ,  $\delta \in \mathcal{B}(\mathbf{R})$ . This implies that the operator  $T^{x^s}$  has not the zero eigenvalue. Take arbitrary marked vectors  $f \in \mathcal{D}$  and  $g \in \mathcal{D}(T^{y^s})$ . Then using the triangle inequality

we deduce that

$$\begin{aligned} | \langle (U^{x^s} - U^{y^s})f; T^{y^s}g \rangle | &= | \langle (U^{x^s} - U^{y^s})T^{y^s}f; g \rangle | = | \langle (U^{x^s}T^{x^s} - U^{y^s}T^{y^s})f; g \rangle \\ &+ \langle U^{x^s}(T^{y^s} - T^{x^s})f; g \rangle | \leq | \langle (B^{x^s} - B^{y^s})f; g \rangle | + \|(T^{y^s} - T^{x^s})f\| \|g\|. \end{aligned}$$

But the limits are zero  $\lim_{x^s \rightarrow y^s} \langle (B^{x^s} - B^{y^s})f; g \rangle = 0$  due to suppositions of this theorem and  $\lim_{x^s \rightarrow y^s} \|(T^{y^s} - T^{x^s})f\| = 0$ , since  $T^x = A^x$  and  $A^x$  has the integral representation given by Formula (9). Thus the limit

$$\lim_{x^s \rightarrow y^s} \langle (U^{x^s} - U^{y^s})f; h \rangle = 0$$

is zero for each  $f \in \mathcal{D}$  and  $h \in \mathcal{R}(T^{y^s})$ . On the other hand,  $\mathcal{D}$  is dense in  $X$ , since  $\bigoplus_{k=1}^{\infty} Y^s$  is dense in  $X$ . The family  $U^x$  of unitary operators is norm bounded by the unit 1, consequently,  $\lim_{x^s \rightarrow y^s} \langle (U^{x^s} - U^{y^s})f; h \rangle = 0$  for each  $f, h \in X$  and hence the semi-group  $\{U^{x^s} : x^s \in \Omega\}$  is weakly continuous. The semi-group  $\{U^{x^s} : x^s \in \Omega\}$  of unitary operators can be extended to a weakly continuous group of unitary operators putting  $U^{-x^s} = (U^{x^s})^*$  and  $U^0 = I$ . This one-parameter commutative group of unitary operators is also strongly continuous, since

$$\begin{aligned} \|(U^{x^s} - U^{y^s})f\|^2 &= \langle (U^{x^s} - U^{y^s})f; (U^{x^s} - U^{y^s})f \rangle = \\ &\langle (U^{x^s} - U^{y^s})^*(U^{x^s} - U^{y^s})f; f \rangle = \langle (2I - U^{x^s-y^s} - U^{y^s-x^s})f; f \rangle \\ &= \langle (U^0 - U^{x^s-y^s})f; f \rangle + \langle (U^0 - U^{y^s-x^s})f; f \rangle. \end{aligned}$$

In view of Theorem I.3.28 [28] there exists a unique  $\mathcal{A}_v$  graded projection valued measure  ${}_s\hat{\mathbf{F}}$  so that

$$(10) \quad \langle U(x^s)f; h \rangle = \int_{-\infty}^{\infty} \langle {}_s\hat{\mathbf{F}}(db_s). \exp(x_s M_s(b_s) b_s) f; h \rangle$$

for each  $f, h \in \mathcal{D}(Q^s)$ , where

$$(11) \quad \langle Q^s f; h \rangle = \int_{-\infty}^{\infty} b_s \langle {}_s\hat{\mathbf{F}}(db_s) f; h \rangle$$

for each  $f, h \in \mathcal{D}(Q^s)$ ,

$$(12) \quad \mathcal{D}(Q^s) = \{f : f \in X; \|Q^s f\|^2 = \int_{-\infty}^{\infty} \langle {}_s\hat{\mathbf{F}}(db_s). b_s^2 f; f \rangle < \infty\},$$

$M_s(b_s)$  is a Borel function from  $\mathbf{R}$  into the purely imaginary unit sphere  $\mathcal{S}_v := \{z \in \mathcal{A}_v : |z| = 1, \text{Re}(z) = 0\}$ . Then we put  ${}_s\hat{\mathbf{E}}(da_s, db_s) = {}_s\hat{\mathbf{E}}(da_s) {}_s\hat{\mathbf{F}}(db_s)$ , where

$${}_s^j\hat{\mathbf{E}}(\delta_1) {}_s^k\hat{\mathbf{F}}(\delta_2) = (-1)^{\kappa(j,k)} {}_s^k\hat{\mathbf{E}}(\delta_1) {}_s^j\hat{\mathbf{F}}(\delta_2)$$

for each  $j, k$  and Borel subsets  $\delta_1, \delta_2 \in \mathcal{B}(\mathbf{R})$ . Then an operator  $P^{x^s}$  exists prescribed by the formula:

$$(13) \quad P^{x^s} = \int_{-\infty}^{\infty} \int_0^{\infty} {}_s\hat{\mathbf{E}}(da_s, db_s) \cdot [a_s^{x^s} \exp(x_s M_s(b_s) b_s)].$$

This implies the inclusion  $B^{x^s} \subseteq P^{x^s}$ , but a normal operator is maximal, consequently,  $B^{x^s} = P^{x^s}$  for each  $s$  and  $x^s \in \Omega$ .

Suppose now that  ${}_s\hat{\mathbf{E}}(\{0\}) \neq 0$ , consider the null space  $N^s := \ker(B^{x^s})$  of  $B^{x^s}$ . To each  $\mathcal{A}_v$  graded projection valued measure  ${}_s\hat{\mathbf{E}}(\delta)$  associated with the family  $\text{alg}_{\mathcal{A}_v}(I, B^x, (B^x)^*)$  a real valued characteristic function in  $\mathcal{N}(\Lambda, \mathbf{R})$  corresponds, where  $\delta \in \mathcal{B}(\mathbf{R}^2)$ , consequently,  $N^s$  is an  $\mathcal{A}_v$  vector subspace in  $X$ . Let  $X = N^s \oplus K^s$ , hence  $K^s$  is an  $\mathcal{A}_v$  vector space, since  $N^s$  is the  $\mathcal{A}_v$  vector subspace of the  $\mathcal{A}_v$  Hilbert space  $X$ . Take the restrictions  $B^{x^s}|_{N^s} =: B^{x^s, N}$  and  $B^{x^s}|_{K^s} =: B^{x^s, K}$  of  $B^{x^s}$  to  $N^s$  and  $K^s$  correspondingly. This implies that the semi-group of normal operators  $\{B^{x^s, K} : x^s \in \Omega\}$  possesses the property that none of the operators  $B^{x^s, K}$  has zero eigenvalue. From Formula (13) it follows, that there exists a two-parameter resolution  ${}_{s,K}\hat{\mathbf{E}}$  of the identity so that

$$(14) \quad B^{x^s, K} = \int_{-\infty}^{\infty} \int_0^{\infty} {}_{s,K}\hat{\mathbf{E}}(da_s, db_s) \cdot [a_s^{x^s} \exp(x_s M_s(b_s) b_s)].$$

Define an  $\mathcal{A}_v$  graded projection value measure  ${}_{s,N}\hat{\mathbf{E}}$  so that

$$\int_{-\infty}^{a_s} \int_{-\infty}^{b_s} {}_{s,N}\hat{\mathbf{E}}(dt_s, dq_s) = {}_{s,N;a_s,b_s}\hat{\mathbf{E}}$$

so that  ${}_{s,N;a_s,b_s}\hat{\mathbf{E}} = 0$  for  $a_s < 0$  and  ${}_{s,N;a_s,b_s}\hat{\mathbf{E}} = I$  when  $a_s \geq 0$ . Since  $B^{x^s, N}(N^s) = \{0\}$ , the integral representation follows:

$$(15) \quad B^{x^s, N} = \int_{-\infty}^{\infty} \int_0^{\infty} {}_{s,N}\hat{\mathbf{E}}(da_s, db_s) \cdot [a_s^{x^s} \exp(x_s M_s(b_s) b_s)].$$

Now it is natural to put  ${}_s\hat{\mathbf{E}}(da_s, db_s) = {}_{s,N}\hat{\mathbf{E}}(da_s, db_s) \oplus {}_{s,K}\hat{\mathbf{E}}(da_s, db_s)$  for an  $\mathcal{A}_v$  graded projection valued measure on  $X$ , that induces the formula:

$$(16) \quad B^{x^s} = \int_{-\infty}^{\infty} \int_0^{\infty} {}_s\hat{\mathbf{E}}(da_s, db_s) \cdot [a_s^{x^s} \exp(x_s M_s(b_s) b_s)].$$



In accordance with Theorem 5

$${}_s^j \hat{\mathbf{E}}(\delta_1) {}_q^k \hat{\mathbf{E}}(\delta_2) = (-1)^{\kappa(j,k)} {}_q^k \hat{\mathbf{E}}(\delta_2) {}_s^j \hat{\mathbf{E}}(\delta_1)$$

for each  $s, q = 1, \dots, n$  and  $j, k = 0, 1, \dots, 2^v - 1$  and every  $\delta_1, \delta_2 \in \mathcal{B}(\mathbf{R}^2)$ , particularly, for  $j = k = 0$  i.e.  ${}_s \hat{\mathbf{E}}(\delta_1)$  and  ${}_q \hat{\mathbf{E}}(\delta_2)$  commute. Then we put

$${}_{(a,b)} \hat{F} = \int_{-\infty}^{a_1} \int_{-\infty}^{b_1} \dots \int_{-\infty}^{a_n} \int_{-\infty}^{b_n} {}_1 \hat{\mathbf{E}}(dt_1, dq_1) \dots {}_n \hat{\mathbf{E}}(dt_n, dq_n),$$

hence  ${}_{(a,b)} \hat{F}$  is an  $\mathcal{A}_v$  graded resolution of the identity, for which

$$d {}_{(a,b)} \hat{F} \cdot \{a^x \exp[x_1 M_1(b_1) b_1] \dots \exp[x_n M_n(b_n) b_n]\} =$$

$${}_1 \hat{\mathbf{E}}(da_1, db_1) \cdot \exp(x_1 M_1(b_1) b_1) \dots {}_n \hat{\mathbf{E}}(da_n, db_n) \cdot \exp(x_n M_n(b_n) b_n) a^x,$$

since the semi-groups  $\{B^x : x \in \Omega\}$  and  $\{T^x : x \in \Omega\}$  and  $\{U^x : x \in \Omega\}$  are commutative, the real field  $\mathbf{R}$  is the center of the Cayley-Dickson algebra  $\mathcal{A}_v$  for each  $v \geq 2$ , the fields  $\mathcal{A}_0 = \mathbf{R}$  and  $\mathcal{A}_1 = \mathbf{C}$  are commutative,  $a_s \in \mathbf{R}$  and  $x_s \in \mathbf{R}$  for each  $s = 1, \dots, n$ . For the operators

$$(17) \quad P^x = \int_{\mathbf{R}^{2n}} d {}_{(a,b)} \hat{F} \cdot \{a^x \exp[x_1 M_1(b_1) b_1] \dots \exp[x_n M_n(b_n) b_n]\},$$

where

$$a^x = \prod_{k=1}^n a_k^{x_k},$$

$M_s : \mathbf{R}^n \rightarrow \mathcal{S}_v := \{z \in \mathcal{A}_v : |z| = 1, \operatorname{Re}(z) = 0\}$  is a Borel function for each  $s$ , the inclusion follows  $B^x \subseteq P^x$  for each  $x \in \Omega$ , since  $B^x = B^{x^1} \dots B^{x^n}$ , where the operators  $B^{x^1}, \dots, B^{x^n}$  pairwise commute. But a normal operator is maximal, consequently,  $B^x = P^x$  for each  $x \in \Omega$ . A uniqueness of the resolution  ${}_{(a,b)} \hat{F}$  of the identity follows from uniqueness of  ${}_s \hat{\mathbf{F}}$  and  ${}_s \hat{\mathbf{E}}$  for each  $s$ .

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